Dynamics of Inflationary Universes with Positive Spatial Curvature

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If the spatial curvature of the universe is positive, then the curvature term will always dominate at early enough times in a slow-rolling inflationary epoch. This enhances inflationary effects and hence puts limits on the possible number of e-foldings that can have occurred, independently of what happened before inflation began and in particular without regard for what may have happened in the Planck era. We use a simple multi-stage model to examine this limit as a function of the present density parameter Ω_0 and the epoch when inflation ends.

I. POSITIVELY-CURVED INFLATIONARY MODELS?

The inflationary universe paradigm [1] is the premiere causative concept in present-day physical cosmology, and faith in this view has been bolstered by the recent measurements of a second and third peak in the cosmic blackbody background radiation (CBR) anisotropy spectrum [2], as has been predicted on the basis of inflationary scenarios. The best-fit models vary according to the prior assumptions made when analyzing the data [3], but together with supernova data [4] suggest a model ($\Omega_{\Lambda_0} \approx 0.7$, $\Omega_{m_0} \approx 0.3$, $\Omega_0 \approx 1$) with a non-zero cosmological constant and sufficient matter to make it almost flat, implying the universe is cosmological-constant dominated from the present back to a redshift of about $z = 0.326$, and matter dominated back from then to decoupling. While the set of models compatible with the data include those with flat spatial sections ($k = 0$), i.e. a critical effective energy density: $\Omega_0 = 1$, they also include positive spatial curvature ($k = +1 : \Omega_0 > 1$) models and negative curvature ($k = -1 : \Omega_0 < 1$) ones, with a weak implication that the best-fit models have positive curvature [3]. This has important implications: if true, it means that the best-fit universe models, extrapolated unchanged beyond the visual horizon, have finite spatial sections and contain a finite amount of matter. Whether they will expand forever or not depends on whether the cosmological constant is indeed constant (when these models will expand forever even though $k = +1$), or varies with time and decays away in the far future (when they will recollapse).

It should be noted that while inflation is taken to predict that the universe is very close to flat at the present time, it does *not* imply that the spatial sections are *exactly* flat; indeed that case is infinitely improbable, and neither inflation nor any other known physical process is able to specify that curvature, nor dynamically change it from its initial value [5]. Thus there is no reason to believe on the basis of inflationary dynamics that $k = 0$. Indeed positive-curvature universes have been claimed to have major philosophical advantages over the flat and negatively curved cases, being introduced first by Einstein [7] in an attempt to solve the problem of boundary conditions at infinity, and then adopted as the major initial paradigm in cosmology by Friedmann, Lemaitre, and Eddington. This view was then taken up, in particular by Wheeler [8], to the extent that the famous book on gravitation he co-authored with Thorne and Misner [9] almost exclusively considered the positive curvature case, labeling the negatively curved case 'model universes that violate Einstein's conception of cosmology' (see page 742). Without going that far, it is certainly worth exploring the properties of inflationary models with $k = +1$ [6], particularly as this case has been marginally indicated by some recent observations.

In this paper we examine the dynamics of inflationary universe models (i) with positive curvature, and (ii) where a cosmological constant approximation holds in the inflationary era, deriving new limits on the allowed numbers of e-foldings of such models as a function of the epoch when inflation ended and of the present-day total energy density parameter Ω_0. These limits do not contradict standard inflationary understanding. Indeed, in a sense they enhance inflation, since early in the inflationary epochs of $k = +1$ universes, the deceleration parameter is *more* negative than in the $k = 0$ models. We only model the Hot Big Bang era (post inflation to the present day) and the Inflationary era; our results are independent of the dynamics before inflation starts. Similar results will hold for all models with only slow rolling inflation. Although the observational evidence is that there is currently a non-zero cosmological constant, as mentioned above, for simplicity we will consider here only the case of an almost-flat $k = +1$ universe with vanishing cosmological constant after the end of inflation. This approximation will not affect the statements derived concerning dynamics up to the time of decoupling, but will make a small difference to estimates of the number of e-foldings given here. We will give more accurate estimates in a more detailed paper on these dynamics [10]. An accompanying paper discusses the implications of this dynamical behaviour for horizons in positively-curved inflationary universes [11].

II. BASIC EQUATIONS

The positive-curvature Friedmann-Lemaître (FL) cosmological model in standard form has a scale factor $S(t)$ normalized so that the spatial metric has unit spatial curvature at the time t_* when $S(t_*) = 1$ (see e.g. [14],[15]). The spatial sections are closed at $r-$ coordinate value increment 2π; that is, $P = (t, r - \pi, \theta, \phi)$ and $P' = (t, r + \pi, \theta, \phi)$ are necessarily the same point, for arbitrary values of t, r, θ, ϕ, and wherever the origin of coordinates is chosen. The Hubble Parameter is $H(t) = \dot{S}(t)/S(t)$, with present value $H_0 = 100h \ km/sec/Mpc$. The dimensionless quantity h probably lies in the range $0.7 < h < 0.5$.

A. $k = +1$ Dynamics

The dynamic behaviour is determined by the Friedmann equation for $k = +1$ FL universes:

$$\left(\frac{H(t)}{c}\right)^2 = \frac{\kappa\mu(t) + \Lambda}{3} - \frac{1}{S(t)^2}, \tag{1}$$

where κ is the gravitational constant in appropriate units and Λ the cosmological constant (see e.g. [14],[15]). The way this works out in practice is determined by the matter content of the universe, whose total energy density $\mu(t)$ and pressure $p(t)$ necessarily obey the conservation equation

$$\dot{\mu}(t) + \left(\mu(t) + p(t)/c^2\right) 3H(t) = 0. \tag{2}$$

The nature of the matter is determined by the equation of state relating $p(t)$ and $\mu(t)$; we will describe this in terms of a parameter $\gamma(t)$ defined by

$$p(t)/c^2 = (\gamma(t) - 1)\,\mu(t), \gamma \in [0, 2]. \tag{3}$$

During major epochs of the universe's history, the matter behaviour is well-described by this relation with γ a constant (but with that constant different at various distinct dynamical epochs). In particular, $\gamma = 1$ represents pressure free matter (baryonic matter), $\gamma = \frac{4}{3}$ represents radiation (or relativistic matter), and $\gamma = 0$ gives an effective cosmological constant of magnitude $\Lambda = \kappa\mu$ (by equation (2), μ will then be unchanging in time). In general, μ will be a sum of such components. However we can to a good approximation represent the universe as a series of simple epochs with only one or at most two components in each epoch.

The dimensionless density parameter $\Omega_i(t)$ for any matter component i is defined by

$$\Omega_i(t) \equiv \frac{\kappa\mu_i(t)}{3}\left(\frac{c}{H(t)}\right)^2 \Rightarrow \Omega_{i0} = \frac{\kappa\mu_{i0}c^2}{3H_0^2} \tag{4}$$

where Ω_{i0} represents the value of $\Omega_i(t)$ at some arbitrary reference time t_0, often taken to be the present time. One can define such a density parameter for each energy density present. We can represent a cosmological constant in terms of an equivalent energy density $\kappa\mu_\Lambda = \Lambda$; from now on we omit explicit reference to Λ, assuming it will be included in this way when necessary. From the Friedmann equation (1), the scale factor $S(t)$ is related to the total density parameter $\Omega(t)$, defined by

$$\Omega(t) = \sum_i \Omega_i(t) = \frac{\kappa\mu(t)}{3}\left(\frac{c}{H(t)}\right)^2 > 1, \tag{5}$$

via the relation

$$\frac{1}{S(t)^2} = \left(\frac{H(t)}{c}\right)^2 (\Omega(t) - 1) \Rightarrow \left(\frac{H_0}{c}\right)^2 = \frac{1}{S_0^2 (\Omega_0 - 1)}, \ \ \Omega_0 \equiv \frac{\kappa\mu_0}{3}\frac{c^2}{H_0^2} > 1, \tag{6}$$

where Ω_0 is the present total value: $\Omega_0 = \sum_i \Omega_{i0}$. When γ is constant for some component labeled i, the conservation equation (2) gives

$$\frac{\kappa\mu_i(t)}{3} = \Omega_{i0}\left(\frac{H_0}{c}\right)^2\left(\frac{S_0}{S(t)}\right)^{3\gamma} = \frac{1}{S_0^2}\left(\frac{\Omega_{i0}}{\Omega_0 - 1}\right)\left(\frac{S_0}{S(t)}\right)^{3\gamma} \tag{7}$$

for each component, on using (6). The integral

$$\Psi(A, B) \equiv \int_{t_A}^{t_B} \frac{cdt}{S(t)} = c \int_{S_A}^{S_B} \frac{dS}{S\dot{S}} \tag{8}$$

is the conformal time, used in the usual conformal diagrams for FL universes [13].

B.   Matter and Radiation Eras

During the *combined matter and radiation eras*, i.e. whenever we can ignore the Λ term in the Friedmann equation (1) but include separately conserved pressure-free matter and radiation: $\mu = \mu_m + \mu_r$, each separately obeying (7), a simple analytic expression relates S and Ψ [15]. For such combined matter and radiation, referred to an arbitrary reference point P in this epoch, we have the exact solution

$$S(\Psi)_P = S_P \left(\frac{1}{2} \frac{\Omega_{m_P}}{\Omega_P - 1} (1 - \cos \Psi) + \sqrt{\frac{\Omega_{r_P}}{\Omega_P - 1}} \sin \Psi \right), \tag{9}$$

where the first term is due to the matter and the second is due to radiation. It is remarkable that they are linearly independent in this non-linear solution. We obtain the pure radiation solution if $\Omega_{m_p} = 0$, and the pure matter solution if $\Omega_{r_p} = 0$. The origin of the time Ψ has been chosen so that an initial singularity occurs at $\Psi = 0$, if the Hot Big Bang era in this model is extended as far as possible (without an inflationary epoch). We will use this representation from the end of inflation to the present day. It will be accurate whenever the matter and radiation are non-interacting in the sense that their energy densities are separately conserved, but inaccurate when they are strongly interacting, for example when pair production takes place.

C.   Cosmological Constant Epoch

During a *cosmological constant-dominated era*, i.e. when $\Lambda > 0$ and we can ignore matter and radiation in (1), we can find the *general solution* (with a suitably chosen origin of time) in the simple collapsing and re-expanding form

$$S(t) = S(0) \cosh \lambda t, \ \lambda \equiv c\sqrt{\frac{\Lambda}{3}}, \ S(0) \equiv \frac{c}{\lambda}, \tag{10}$$

where $t = 0$ corresponds to the minimum of the radius function, i.e. the turn-around from infinite collapse to infinite expansion, and so $S(0)$ is the minimum value of $S(t)$ (note that we can have independent time scales in the different eras with different zero-points, provided we match properly between eras as discussed next). This is of course just the de Sitter universe represented as a Robertson-Walker spacetime with positively-curved space sections [18] , and can be used to represent an inflationary universe era for models with $k = +1$ if we restrict ourselves to the expanding epoch

$$t \geq t_i \geq 0 \tag{11}$$

for some suitable initial time t_i which occurs after the end of the Planck era, so $t_i \geq t_{Planck}$. We will represent the inflationary era (preceding the Hot Big Bang era) in this way.

The Hubble parameter is $H(t) = \lambda \tanh \lambda t$ (zero at $t = 0$ and positive for $t > 0$) and the density parameter $\Omega_\Lambda(t)$ is given by

$$\Omega_\Lambda(t) = \frac{\Lambda}{3} \frac{c^2}{H^2(t)} = \frac{1}{(\tanh \lambda t)^2}, \tag{12}$$

which diverges as $t \to 0$ and tends to 1 as $t \to \infty$. The inflationary effect is enhanced in such models as compared with $k = 0$ models, because the deceleration parameter $q = -\ddot{S}/(SH^2)$ is here even more negative than in those scale-free models.

D. Joining Different Eras

Junction conditions required in joining two eras with different equations of state are that we must have $S(t)$ and $\dot{S}(t)$ continuous there, thus $H(t)$ is continuous also. By the Friedmann equation this implies in turn that $\mu(t)$ is continuous, so by its definition $\Omega(t)$ is also continuous (note that it is $p(t)$ that is discontinuous on spacelike surfaces of discontinuity). We need to demand, then, that any two of these quantities are continuous where the equation of state is discontinuous; for our purposes it will be convenient to take them as $S(t)$ and $\Omega(t)$. Thus we need to know $S(t)$ and $\Omega(t)$ at the beginning and end of each era to get a matching with $S_- = S_+$ and $\Omega_- = \Omega_+$.

The matching we need to perform is between the Hot Big Bang era and the Inflationary era. Now for combined matter and radiation, referred to an arbitrary reference point P, we have (9). Writing the same solution in the same form (with the initial singularity at $\Psi = 0$ in both cases) but referred to another reference point Q, we have the same expressions but with P replaced by Q everywhere. As these are the same evolutions referred to different events, $S(\Psi)_P = S(\Psi)_Q$ for all Ψ, so they must have identical functional forms. Matching the two expressions for all Ψ, the coefficients for $\cos\Psi$ and $\sin\Psi$ on each side must separately be equal. Letting $S_P/S_Q = \mathcal{R}$, this gives the total density parameter $\Omega_Q(\mathcal{R}) = \Omega_{mQ}(\mathcal{R}) + \Omega_{rQ}(\mathcal{R})$ at the event Q in terms of the density parameter values at P. Taking the event Q to be the end of inflation and the event P to be here and now given by $t = t_0$, we find

$$\Omega_Q(\mathcal{R}) = \frac{\mathcal{R}\ (\Omega_{m0} + \mathcal{R}\Omega_{r0})}{\mathcal{R}^2\Omega_{r0} + \mathcal{R}\ \Omega_{m0} - (\Omega_{m0} + \Omega_{r0} - 1)}. \tag{13}$$

The matching condition is then given by assuming $\Omega_\Lambda(t_Q) = \Omega_Q(\mathcal{R})$. We are assuming here that the details of reheating at the end of inflation are irrelevant: conservation of total energy must result in the total value of Ω at Q being constant during any such change (one can easily modify this condition if desired). This gives $\Omega_\Lambda(t_Q)$ at the end of inflation Q by (12), which then gives t_Q in the inflationary epoch described by (10):

$$t_Q = \frac{1}{\lambda}\text{arctanh}\frac{1}{\sqrt{\Omega_\Lambda(t_Q)}} = \frac{1}{\lambda}\text{arctanh}\frac{1}{\sqrt{\Omega_Q(\mathcal{R})}}\ , \tag{14}$$

with $\Omega_Q(\mathcal{R})$ given by (13). Note that in these expressions, the ratio $\mathcal{R} = S_0/S_Q$ is the expansion ratio from the end of inflation until today.

III. PARAMETER LIMITS

A. Maximum Number of e-foldings: $N_{\max}$

The maximum number of e-foldings available until time t_Q with the given value $\Omega_\Lambda(t_Q)$ is given by the expansion form (10) with $t = t_Q$ given by (14):

$$e^{N_{\max}} = \frac{S(t_Q)}{S(0)} = \cosh\left(\text{arctanh}\frac{1}{\sqrt{\Omega_Q(\mathcal{R})}}\right), \tag{15}$$

because $S(0)$ is the minimum value of $S(t)$ (by our choice of time coordinate for this era, the throat where expansion starts is set at $t = 0$) and t is restricted by (11). If the universe starts off at any time later than $t = 0$ in the expanding era $t > 0$, there will be fewer e-foldings before the end of inflation t_Q.

Defining the difference of Ω_0 from unity to be $\delta > 0$:

$$\Omega_0 = 1 + \delta \Leftrightarrow \Omega_{m0} = 1 + \delta - \Omega_{r0}\ , \tag{16}$$

we find from (15) and (13) that the maximum number of inflationary e-foldings that can occur before inflation ends at an event Q with expansion ration $\mathcal{R}$, is

$$N_{\max}(\mathcal{R}, \delta) = \ln\left[\cosh\left(\text{arctanh}\frac{\sqrt{\alpha - \delta}}{\sqrt{\alpha}}\right)\right] = \frac{1}{2}\ln\alpha - \frac{1}{2}\ln\delta\ , \tag{17}$$

where

$$\alpha \equiv \mathcal{R}^2\Omega_{r0} + \mathcal{R}(1 + \delta - \Omega_{r0}).$$

This e-folding limit essentially represents a matching of the present day radiation density Ω_{r0} to the energy density limits that may be imposed at the end of inflation, which will place restrictions on the possible value of the expansion ratio $\mathcal{R}$. It does not take into account matter-radiation conversions in the Hot Big Bang era, which we consider in a later section.

B. Density parameter variation from unity: δ

What are the implications? The inversion of (17) in terms of δ is

$$\delta(\mathcal{R}, N_{\max}) = \mathcal{R}\frac{\mathcal{R}\Omega_{r0} + 1 - \Omega_{r0}}{e^{2N_{\max}} - \mathcal{R}}. \tag{18}$$

Note that this value diverges when $N_{\max} = \frac{1}{2}\ln(\mathcal{R})$, so this is the value corresponding to a turn-around today ($\Omega_0 = \infty \Leftrightarrow H_0 = 0$). Thus there is a smallest value for $N_{\max}$ for each set of parameters $\Omega_{r0}, \mathcal{R}$.

C. Expansion ratio since end of inflation: $\mathcal{R}$

The inversion in terms of the ratio $\mathcal{R}$ is

$$\mathcal{R}(\delta, N_{\max}) = \frac{1}{2\Omega_{r0}}\left(\sqrt{4\Omega_{r0}\delta e^{2N_{\max}} + (1 + \delta - \Omega_{r0})^2} - (1 + \delta - \Omega_{r0})\right). \tag{19}$$

For large $\delta e^{2N_{\max}}$ this is well-approximated by

$$\mathcal{R}(\delta, N_{\max}) = \frac{1}{\sqrt{\Omega_{r0}}}\sqrt{\delta}e^{N_{\max}}. \tag{20}$$

D. Numerical Values

The epoch chosen for the end of inflation will determine the expansion parameter $\mathcal{R}$. What is a realistic expectation for the end of inflation? A typical figure for the energy then is 10^{14}Gev, just below the GUT energy. In terms of temperature this is equivalent to $T = 1.16 \times 10^{27}K$ at the end of inflation. But the CBR temperature is $2.75K$ today, so assuming that in the Hot Big Bang era T scales as $1/S(t)$, we obtain the value $\mathcal{R} = 1.16 \times 10^{27}/2.75 = 4.22 \times 10^{26}$. This can be taken as an upper value (inflation ends below the GUT energy), but requires correction for pair production processes at high temperatures (see below). An absolute lower value would be $\mathcal{R} = 10^{12}$ (ensuring that inflation ends before baryosynthesis and nucleosynthesis begin). Finally, how many e-foldings would be expected during inflation? A value demanded in most inflationary scenarios is at least $N = 60$, required firstly to smooth out the universe, and then assumed in the usual structure formation studies. A typical figure is an expansion ratio $e^N = e^{70} = 2.5 \times 10^{30}$ ([19], p.355); some studies quote much higher values for N. The value of the difference from flatness δ today (16) is probably in the range $-0.05 < \delta < 0.1$; it might be very small indeed, as assumed in many inflationary scenarios. In this paper, we are only studying the case $\delta > 0$ because we are assuming positive spatial curvature. The Cosmic Background Radiation density today is well-determined from its temperature of $T = 2.75K$, and is $\Omega_{r0} = 4.2 \times 10^{-5}h^{-2}$ [19], because we include the neutrino degrees of freedom here. Taking $h = 0.65$, this gives the value $\Omega_{r0} \simeq 10^{-4}$.

We now explore the effect of variation of all these parameters except Ω_{r0}, which we take as fixed, because the temperature of that radiation is extremely well determined by observation. It is this quantity that then determines the numbers in what follows (if we did not fix this number, we would get only functional relations but not specific numerical limits on what can happen). There will be a small variation in the results that follow if we vary h, because the CBR temperature is converted into an equivalent Ω_{r0} value by the present value of the Hubble constant (expressed in terms of h).

E. Allowed end of inflation

In terms of the ratio $\mathcal{R}$, for $\Omega_0 = 1 + \delta$ and using the above value for Ω_{r0}, we can get at most N e-foldings during inflation if the end of inflation occurs at the expansion ratio

$$\mathcal{R}(\delta, N_{\max}) = \frac{1}{8.4 * 10^{-5}h^{-2}}\left(\sqrt{16.8 * 10^{-5}h^{-2}\delta e^{2N_{\max}} + \Delta^2} - \Delta\right), \tag{21}$$

$$\Delta \equiv (1 + \delta - 4.2 \times 10^{-5}h^{-2}).$$

For $N_{\max} \geq 60$ and $\delta > 10^{-4}$ we can use the simple approximation

$$\mathcal{R}(\delta, N_{\max}) = 154.3h\sqrt{\delta}e^{N_{\max}}. \tag{22}$$

Values for $N_{\max} = 60$ and $h = 0.65$ are

δ	0.0001	0.0005	0.0008	0.001
$\mathcal{R}$	1.45×10^{26}	2.56×10^{26}	3.24×10^{26}	3.62×10^{26}
δ	0.01	0.1	1	2
$\mathcal{R}$	1.15×10^{27}	3.62×10^{27}	1.46×10^{28}	1.62×10^{28}
δ	3	4	5	10
$\mathcal{R}$	1.98×10^{28}	2.29×10^{28}	2.56×10^{28}	3.62×10^{28}

Now as commented above, we do not want to exceed the value $\mathcal{R} = 4.22 \times 10^{26}$ corresponding to the GUT energy density. The conclusion is that we exceed this value if $\delta > 0.005$. The limit will become stronger if we demand more e-foldings.

F. Allowed density range today

Assume now $\mathcal{R} = 4.22 \times 10^{26}$ as in standard texts [19]. Then, neglecting a small term we obtain

$$\delta(N_{\max}) = \frac{7.47 \times 10^{48}}{h^2 \left(e^{2N_{\max}} - 4.22 \times 10^{26}\right)} .$$

In this case, the smallest number $N_{\max}$ is given by $e^{2N_{\max}} = 4.22 \times 10^{26}$, and so $N_{\max} > 30.65$. For various inflationary e-folding values $N_{\max}$ greater than this amount, we find, on setting $h = 0.65$:

$N_{\max}$	40	50	55	56
δ	3.19×10^{14}	6.58×10^{5}	29.86	4.04
$N_{\max}$	57	58	59	60
δ	0.55	7.40×10^{-2}	1.00×10^{-2}	1.36×10^{-3}

We see here the very sharp decline as $N_{\max}$ increases through 56 to 59. Values higher than 58 strongly limit the value of δ, i.e. the allowed domain in the $(\Omega_\Lambda, \Omega_M)$ plane.

G. Allowed e-foldings

Finally again assuming $\mathcal{R} = 4.22 \times 10^{26}$, the maximal number of e-foldings is given by

$$N_{\max}(\delta) = 30.654 + \frac{1}{2}\ln\left(1.7724 \times 10^{22} + h^2 + \delta h^2\right) - \ln h - \frac{1}{2}\ln\delta . \tag{23}$$

So for various values of δ, if $h = 0.65$, we find the allowed number of e-foldings:

δ	0.00001	0.0001	0.001
N_{max}	62.46	61.31	60.15
δ	0.01	0.1	1
$N_{\max}$	59.00	57.85	56.70
δ	2	4	10
$N_{\max}$	56.35	56.01	55.55

So we again see the here the crucial e-folding range 56 to 58 as the limit allowing substantial values of δ. This range is less than that normally assumed for the end of inflation.

IV.   ACTUAL NUMBER OF E-FOLDINGS: N

The actual number of e-foldings during the inflationary era until time t_Q, with the Ω-value $\Omega_\Lambda(t_Q)$, starting from time t_i, with the $\Omega-$value $\Omega_\Lambda(t_i)$, is

$$e^N = \frac{S(t_Q)}{S(t_i)} = \frac{\cosh\left(\operatorname{arctanh}\frac{1}{\sqrt{\Omega_Q}}\right)}{\cosh\left(\operatorname{arctanh}\frac{1}{\sqrt{\Omega_i}}\right)} = \sqrt{\frac{\Omega_i - 1}{\Omega_i}}\sqrt{\frac{\Omega_Q}{\Omega_Q - 1}}, \tag{24}$$

(from (10,12)), that is

$$N(\Omega_Q, \Omega_i) = \frac{1}{2}\ln\left\{\frac{\Omega_Q\,(\Omega_i - 1)}{\Omega_i\,(\Omega_Q - 1)}\right\}. \tag{25}$$

The solution for Ω_Q is

$$\Omega_Q(N, \Omega_i) = \frac{e^{2N}\Omega_i}{e^{2N}\Omega_i - \Omega_i + 1},$$

so using (13), substituting $\Omega_0 = 1 + \delta$, and solving for δ gives:

$$\delta(\mathcal{R}, N, \Omega_i) = \frac{\mathcal{R}\left\{\mathcal{R}\Omega_{r0} + 1 - \Omega_{r0}\right\}(\Omega_i - 1)}{e^{2N}\Omega_i - \mathcal{R}(\Omega_i - 1)}. \tag{26}$$

This gives the standard result that inflation through N e-foldings decreases δ, and can indeed make it arbitrarily small if N is large enough. The limit giving $N_{\max}$ is the irregular limit: $\Omega_i \longrightarrow \infty$. We obtain a minimum allowed number of e-foldings from the requirement that $e^{2N}\Omega_i > \mathcal{R}(\Omega_i - 1)$. This gives $N_{\min}$ (the value when δ diverges) to be

$$N_{\min}(\mathcal{R}, \Omega_i) = \frac{1}{2}\ln\left(\frac{\mathcal{R}(\Omega_i - 1)}{\Omega_i}\right). \tag{27}$$

Universes with less e-foldings will have collapsed before today.

V.   IMPLICATIONS

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We have arrived at the following interesting result: Consider a universe with a *cosmological constant dominated inflationary epoch,* where inflation ends by $10^{14}GeV$. Then, noting that $\Omega_0 > 1 \Rightarrow k = +1$, we find that with our assumptions above, if $\Omega_0 > 1.01$, the limits above apply in our multi-stage simple model and there cannot have been inflation through 60 e-foldings or more in such a model. Thus for example $\Omega_0 = 1.01$ contradicts the possibility of an exponentially expanding inflationary scenario with more than 60 e-foldings in our past in such a model. This is essentially because the curvature enhances the effect of inflation in the very early universe, making the curve $S(t)$ bend up more than it would have done in the zero-curvature case and resulting in Ω diverging at a turn-around point if the inflationary era is extended too far to the past. This is disallowed by the instability of a collapsing inflationary epoch [20].

However these values depend on the assumptions we make for $\mathcal{R}$ and h in this simple multi-stage model, and would be changed by more accurate models; there will be variations of these figures with detailed inflationary scenarios and more accurate modeling. In particular, we have carried out preliminary estimates of the effects of (a) changing matter-radiation relations in the hot big bang era, due to pair creation and extra degrees of freedom arising; these seem to make little difference; and (b) the effect of a previous radiation dominated era at the start of inflation, resulting in an initial inflationary era where radiation was non-negligible. The basic effect would remain in this case, but the numbers estimated above would change. These refinements will be considered in a paper [10] examining the relevant dynamics in more detail.

The main point of this paper is that such limits exist and should be taken into account when examining inflationary models with $k = +1$. The limits given above are only for the simple model considered here; they will be different in more detailed models.

A.   Criterion for this to happen

This calculation is for an epoch of inflation driven by a cosmological constant. However there are numerous other forms of inflation. The key point then is that similar effects will occur in all inflationary models in which the effective energy density of the scalar field varies more slowly than the curvature term in the Friedmann equation, which varies as S^{-2}. From (7), this will happen if $3\gamma < 2$. The limiting behaviour where the energy density mimics the curvature term is a coasting universe with $3\gamma = 2 \Leftrightarrow \mu + 3p/c^2 = 0$. Scalar fields can give any effective γ from 0 to 2, so there will be fast-rolling scalar-field driven models where $3\gamma > 2$. However these will not then be inflationary, for they will not be accelerating (the requirement for an accelerating universe is $\mu + 3p/c^2 < 0$). Thus *effects of the kind considered here will occur in all positive curvature inflationary universes*, but power-law models will have different detailed behaviour than the ones with an effective cosmological constant calculated above. The numbers will be different and the constraints may be much less severe.

VI.   CONCLUSION

If we ever *observationally determine* that $\Omega_0 > 1 \Rightarrow k = +1$, then $\delta > \delta_0$ where δ_0 is some value sufficiently large that we can distinguish the value of Ω_0 from unity, and so will certainly be greater than 0.01 (for otherwise we could not observationally prove that $\Omega_0 > 1$). Thus there cannot in this case have been exponential inflation through some value that will depend on the model used; in the case considered above, it is about 59 e-foldings, so such an inflationary scenario, with 60 or more e-foldings, could not have occurred. Hence it is of considerable interest to try all forms of cosmological tests to determine if $\Omega_0 > 1$. It is of course possible we will never determine observationally whether $\Omega_0 > 1$ or $\Omega_0 < 1$. The point of this paper is to comment that there are substantial dynamical implications if we can ever make this distinction on the basis of observational data. There is not a corresponding implication on the negative side, i.e. for $\delta < 0 \Leftrightarrow \Omega_0 < 1$ (one might then claim limits on the number of e-foldings caused by limits on Ω_{Planck} or H_{Planck} at the end of the Planck time; but the results presented here are independent of any such considerations). Thus if we could ever determine say that $\Omega_0 = 0.99$, this would not imply any limit on the number of e-foldings, whereas for $\Omega_0 = 1.01$, such restrictions are implied.

Many inflationary theorists would not find this conclusion surprising, as they would expect the final value of δ to be very small, as is indicated here, and would assume that if we were too far from flat today this was just because, given the starting conditions for the inflationary era, one had not had enough e-foldings to truly flatten the universe; so more e-foldings should be employed, and we would end up much closer to flat today. However they have arrived at that conclusion by examining the case of scale-free (exponential) expansion, which arises when the spatial curvature term in the Friedmann equation is ignored, and then placing bounds on the value of the allowed energy density at the start of inflation. But the point of the present analysis is precisely that one cannot ignore that curvature term at early enough times in an inflationary epoch driven by a cosmological constant. It is the resulting non-scale-free behaviour that leads to the restrictions on allowed e-foldings calculated above, irrespective of the initial conditions inherited from the Planck era. The implication is that if you call up the extra e-foldings needed for that programme just outlined, and end up consistent with the presently observed CBR density, then necessarily a limit such as $\Omega_0 < 1.001$ holds. Thus this kind of result strengthens the inflationary intuition. However that e-folding limit is not incorporated in the models usually used to calculate the CBR anisotropy.

Indeed if $\Omega_0 > 1$, so that only restricted e-foldings can occur and be compatible with the observed CBR temperature, this could have significant effects on structure formation scenarios. The usual analyses resulting in the famous observational planes with axes Ω_m and Ω_Λ [3] are based on assuming that more than 60 e-foldings can occur even if $k = +1$. We suggest the theoretical results need re-examination in the domain where $k = +1$ and only a restricted number of e-foldings can occur. The major point is that the dynamical behaviour is discontinuous in that plane: as Ω_0 varies from $1 - \delta$ to $1 + \delta$, however small δ is, the curvature sign k changes from -1 to $+1$ and the corresponding term $k/S(t)^2$ in the Friedmann equation - which necessarily dominates over any constant term in that equation, for small $S(t)$ - completely changes in its effects. When $k = +1$ it eventually causes a turn-around for some t_0; when $k = -1$ it hastens the onset of the initial singularity.

It should be noted that this conclusion is based purely on examining inflation in FL universe models with a constant vacuum energy, and is not based on examinations of pre-inflationary or Trans-Planckian physics on the one hand, nor on studies of embedding such a FL region in a larger region on the other. It is based solely on the dynamics during the inflationary epoch. However it considers only a constant vacuum energy, equivalent to a no-rolling situation, and so does not take scalar field dynamics properly into account. It will be worth examining slow-rolling and fast-rolling models to see what the bounds of behaviour for $k = +1$ inflationary models are in those cases. We indicated above that insofar as these universes are inflationary (i.e. they are accelerating during the scalar-field dominated era), similar e-folding bounds may be expected in these cases also. Also as indicated above, the results will be modified if there is

a substantial radiation density during the initial phase of inflation. We are currently investigating the difference that this will make.

We are fully aware that in order to properly study the issue, we need to examine anisotropic and inhomogeneous geometries rather than just FL models, because analyses based on FL models with their Robertson-Walker geometry cannot be used to analyse very anisotropic or inhomogeneous eras. Nevertheless this study shows there are major dynamical differences in inflationary FL universes with $k = +1$ or $k = 0$. The implication is (a) that we need to try all observational methods available to determine if $k = +1$, because this makes a significant difference not only to the spatial topology, but also to the dynamical and causal structure of the universe, and (b) we should examine inhomogeneous inflationary cosmological models to see if any similar difference exists between models that are necessarily spatially compact, and the rest.

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